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STRUCTURAL TRANSFORMATIONS OF NON-CONSERVATIVE SYSTEMS†

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The method of structural mappings of gyroscopic systems [1, 2] is developed for systems involving non-conservative positional forces. This technique, considered in the aspect of the legitimate use of the precessional equations of the applied theory of gyroscopes, enables the difficulties associated with the presence of non-conservative structures in the initial equations to be overcome, and in many cases enables of the Thomson–Tait–Chetayev theorems to be used directly. © 2001 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

We will investigate a matrix equation of the form

$$J\ddot{x}+(D+HG)\dot{x}+(\Pi+P)x=X(x,\dot{x}) \tag{1.1}$$

where $x = \mathrm{col}(x_1, \ldots, x_n)$ is an unknown vector, $J = J^T$, $D = D^T$, $G = -G^T$, $\Pi = \Pi^T$, $P = -P^T$ (the superscript T denotes transposition) are constant $n \times n$ matrices, $X(x, \dot{x})$ is an n-dimensional column vector containing x and $\dot{x}$ to powers greater than 1 and $H > 0$ is a certain large scalar parameter. It will be assumed throughout that the matrices J and D are positive-definite and the matrices G and P non-singular; accordingly, n must be even.

Equation (1.1) describes the perturbed motion of many dynamical systems driven by dissipative, gyroscopic, potential and non-conservative positional forces. We will assume throughout that

$$J=\mathrm{diag}\,(J_1,\ldots,J_n),\qquad D=\mathrm{diag}\,(b_1,\ldots,b_n) \tag{1.2}$$

In systems containing gyroscopes, J will be the matrix of total moments of inertia about the relevant axes.

Together with Eq. (1.1) we will consider an equation which is usually referred to as the precessional equation. It is derived from (1.1) by neglecting the first matrix term:

$$(D+HG)\dot{u}+(\Pi+P)u=U(u,\dot{u}) \tag{1.3}$$

Mechanically interpreted, the replacement of the initial equation by the approximate equation (1.3) essentially means that allowance is made in the total kinetic energy of the system (or in the total angular momentum) only for the part due to the rotation of the gyroscopes, which is assumed to be fast. Under such conditions the system obtained is half the order of the original one, which, of course, is extremely convenient in the practical application of precession theory.

However, the transition to the equations of precession theory requires suitable justification, since it is not always legitimate. The mathematical basis for the admissibility of using Eq. (1.3) has been studied by many researchers, who have considered both the mathematical aspect of the problem and the construction and development of practical methods of solution [3–9]. In that connection we draw attention, for example, to the methods of integral transformations and splitting transformations, which lead to the classification of motions as slow (precessional) and fast (nutational).

A serious obstacle to the formal transition to the equations of precession theory is the presence in the initial equations of non-conservative positional terms. In that case, an asymptotically stable solution obtained via precession theory may turn out to be unstable in the exact equations, owing to divergence of fast nutational motions [3, 9]. This is further complicated by the fact that the presence of non-conservative positional structures in the equations of perturbed motion of gyroscopic systems excludes the use of the Thomson–Tait–Chetayev theorems [10].

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The theory set out below, unlike the method of splitting transformations, does not require the use of asymptotic expansions, being based on a structural transformation of Eq. (1.1) [1, 2], which enables the matrix of non-conservative positional forces to be eliminated from the transformed equation.

2. TRANSFORMATION OF EQ. (1.1)

We will change to a new variable in (1.1) by setting

$$x = L\xi \tag{2.1}$$

where the matrix L will be determined later. This transformation – the inverse of that used in [1, 2] – turns out to be more convenient in the present case. As a result, we obtain the equation

$$JL\ddot{\xi} + [2J\dot{L} + (D + HG)L]\dot{\xi} + [J\ddot{L} + (D + HG)\dot{L} + (\Pi + P)L]\xi = \Xi \tag{2.2}$$

where Ξ is a column vector containing ξ and $\dot{\xi}$ in powers higher than unity.

A part from the non-linear vector Ξ, Eq. (2.2) may be written as

$$\ddot{\xi} + L^{-1}[2\dot{L} + J^{-1}(D + HG)L]\dot{\xi} + L^{-1}[\ddot{L} + J^{-1}(HG\dot{L} + \Pi L + D\dot{L} + PL)]\xi = 0 \tag{2.3}$$

satisfaction of the condition

$$D\dot{L} + PL = 0 \tag{2.4}$$

enables us to eliminate the matrix P from Eq. (2.3). Condition (2.4), which is similar to the second of conditions (2.6) in [2], may be expressed as a matrix equation

$$\dot{L} = AL \quad (A = D^{-1}P^T) \tag{2.5}$$

whose solution is

$$L = e^{At}L(0) \tag{2.6}$$

where $L(0)$ corresponds to the starting time.

If $L(t)$ and $\dot{L}(t)$ are bounded in the interval $[0, \infty)$, and in addition $|\det L(t)| \geqslant m > 0$ (where m is a positive constant), then L will be a Lyapunov matrix. In that case transformation (2.1) does not change the stability properties of the linear part of the initial equation (1.1).

Taking Eq. (2.4) into consideration, we replace $\dot{L}$ and $\ddot{L}$ in Eq. (2.3), after which the latter may be written in the form

$$\ddot{\xi} + Q\dot{\xi} + R\xi = 0 \tag{2.7}$$

where

$$\begin{gathered} Q = L^{-1}VL, \quad R = L^{-1}WL \\ V = J^{-1}(D + HG) + 2A, \quad W = A^2 + J^{-1}(\Pi + HGA) \end{gathered} \tag{2.8}$$

Thus, the matrices V and Q, and accordingly W and R also satisfy a similarity relation.

The general equation (2.7) will be the basis for the following analysis.

3. ADJUSTABLE GYROSCOPIC SYSTEMS

It is will known that a real skew-symmetric matrix may always, by means of an orthogonal similarity transformation, be reduced to canonical block-diagonal form. In that connection we note that there is an extensive class of dynamical systems described by Eq. (1.1) in which the skew-symmetric matrices G and P are represented from the start in block-diagonal form, with only 2×2 blocks along the principal diagonal

$$G = \operatorname{diag}(G_1, \ldots, G_q), \quad P = \operatorname{diag}(P_1, \ldots, P_q), \quad 2q = n$$
$$G_k = g_k S, \quad P_k = p_k S, \quad S = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} \tag{3.1}$$
$$g_k > 0, \quad p_k > 0, \quad k = 1, 2, \ldots, q$$

Such systems include, in particular, platform poly-gyroscopic gyrohorizons equipped with control of the radial correction type.

Taking representation (3.1) into consideration, we return to Eq. (2.5), putting $L = \|l_{ij}\|_1^n$. We have

$$\|\dot{l}_{ij}\|_1^n = \operatorname{diag}(b_1^{-1}, \ldots, b_n^{-1}) \operatorname{diag}(-P_1, \ldots, -P_q) \|l_{ij}\|_1^n \tag{3.2}$$

Using (3.2), we arrive at equations in the elements l_{ij} of the matrix L, which fall into q groups of independent equations

$$\ddot{l}_{ij} + \Omega_k^2 l_{ij} = 0, \quad i, j = 1, 2, \ldots, n; \quad k = 1, 2, \ldots, q \tag{3.3}$$

Letting the solutions of Eqs (3.3) satisfy individual initial conditions, we obtain the solution (2.6) for the class of dynamical systems under consideration, in the form

$$L = \operatorname{diag}(L_1, \ldots, L_q) \tag{3.4}$$

where

$$L_k = \begin{Vmatrix} \cos\Omega_k t & -r_k^{1/2} \sin\Omega_k t \\ r_k^{-1/2} \sin\Omega_k t & \cos\Omega_k t \end{Vmatrix}$$
$$r_k = b_{2k} / b_{2k-1}, \quad \Omega_k = p_k (b_{2k-1} b_{2k})^{-1/2}, \quad k = 1, 2, \ldots, q \tag{3.5}$$

In view of representations (3.5), if $b_1 = b_2 = \ldots b_n$, the Lyapunov matrix (3.4) becomes orthogonal.

Let us assume that the forces acting on the system, which are modelled by the matrix D in Eq. (1.1), are due solely to a resistant medium. In that case the total moment of the drag forces (dissipation is assumed to be complete) depends, generally speaking, not only on the physical properties of the medium but also on the mass distribution in the system. Following the Sommerfeld–Greenhill conceptions as applied to the system under consideration, we assume that

$$D = \mu J \tag{3.6}$$

where $\mu > 0$ is a small constant scalar parameter, depending on the properties of the medium, and J is the matrix of moments of inertia of the system, defined by the first formula of (1.2) [11].

Taking the foregoing reasoning into consideration, we return to the last two formulae of (2.8), bearing in mind Eqs (1.2), (3.1) and (3.6). We have

$$V = \mu E_n + \operatorname{diag}\left(\begin{Vmatrix} 0 & \sigma_1 \\ -\sigma_2 & 0 \end{Vmatrix}, \ldots, \begin{Vmatrix} 0 & \sigma_{n-1} \\ -\sigma_n & 0 \end{Vmatrix} \right) \tag{3.7}$$
$$\sigma_{2k-1} = (\mu J_{2k-1})^{-1}(\mu H g_k - 2p_k), \quad \sigma_{2k} = (\mu J_{2k})^{-1}(\mu H g_k - 2p_k)$$
$$W = \operatorname{diag}(\chi_1 E, \ldots, \chi_q E) + J^{-1}\Pi$$
$$\chi_k = \mu^{-2}(J_{2k-1}J_{2k})^{-1} p_k (H g_k \mu - p_k), \quad k = 1, 2, \ldots, q$$

where E_n is the identity matrix of order n; when $n = 2$ the subscript is omitted.

A mechanical system driven solely by non-conservative forces will always be unstable, and moreover it will be independent of the higher-order terms represented in Eq. (1.1) by the vector $X(x, \dot{x})$. We can nevertheless ensure asymptotic stability in a non-conservative system in the case when, besides non-conservative positional forces, dissipative and gyroscopic forces are also present, and the number of coordinates determining instability is even [3]. The apparatus described above enables us to use this fact in a legitimate transition to the equations of precession theory, as applied to systems in which conditions (3.1) are satisfied.

Setting $\Pi = 0$ in the last formula of (2.8) and taking (3.7) into account, we verify that the conditions $Q = V$ and $R = W$ hold identically in expressions (2.8). Multiplying Eq. (2.7) on the left by the matrix J, we now have

$$J\ddot{\xi} + V_1\dot{\xi} + W_1\xi = 0 \tag{3.8}$$

where it must be assumed that

$$\begin{aligned} &V_1 = JV = D + \operatorname{diag}(h_1 S, \ldots, h_q S) \\ &h_k = \mu^{-1}(H\mu g_k - 2p_k) \\ &W = JW = \operatorname{diag}(c_1, \ldots, c_n) \\ &c_{2k-1} = \chi_k J_{2k-1}, \quad c_{2k} = \chi_k J_{2k}, \quad k = 1, 2, \ldots, q \end{aligned} \tag{3.9}$$

Thus, the matrix W_1 in Eq. (3.8) has turned out to be diagonal, and it therefore has no non-conservative structures. The Thomson–Tait–Chetayev theorems are thus applicable to Eq. (3.8).

Indeed, if it is assumed that there are no dissipative and gyroscopic terms in the matrix V_1 of (3.9), quantities c_i (which are the Poincaré coefficients for this case [11]) will be positive, as is implied by the inequalities

$$Hg_k\mu - p_k > 0, \quad k = 1, 2, \ldots, q \tag{3.10}$$

which follow from formulae (3.7) and (3.9). These correspond to (non-asymptotic) stability of the trivial solution of Eq. (3.8) under the specified conditions. If inequalities (3.10) hold then, by the corresponding Thomson–Tait–Chetayev theorem, the addition of drag forces with complete dissipation and arbitrary gyroscopic forces (corresponding to $h_k \neq 0$) will make Eq. (3.8) asymptotically stable and accordingly nutational oscillations will be damped out.

Inequalities (3.10) yield lower bounds for the dissipation coefficient μ

$$\mu > \max_{1 \leq k \leq q} (p_k / Hg_k) \tag{3.11}$$

which guarantee damping of nutation. Inequalities (3.10) agree with the sufficient condition for stability obtained previously for the case in which $\|p_{kj}\| = \alpha H \|g_{kj}\|$, where α is a positive constant, but without involving the problem of changing to the equations of precession theory. Indeed, setting $p_k = \alpha H g_k$ in (3.1), we obtain the condition $\mu > \alpha$ [12].

If $\Pi = 0$, the structures of matrices (3.1) and (3.4) imply that Eqs (1.1) and (3.10) have q independent groups of second-order scalar equations.

If the system involves potential forces ($\Pi \neq 0$ in (1.1)), the conditions for damping nutational oscillations may be insufficient to guarantee stability of the entire system, although they must be stipulated in any case. On the assumption that the parameter H is sufficiently large, one must also guarantee asymptotic stability in precessional equation (1.3) [3].

We will now consider some illustrative examples.

4. A SINGLE-ROTOR ADJUSTABLE GYROCOMPASS

The selective property of a gyroscopic compass relative to the meridian direction of a location is achieved through a pendulum effect, which acts in such a way that the intrinsic angular momentum vector of the compass is fixed in the meridian plane of the location. The pendulum effect may be realized in various ways, say be attaching a special pendulum to the sensitive element.

Adjustable course indicators based on an astatic three-stage gyroscope are widely used at present. In such course indicators the torques necessary to generate the pendulum and damping effects are established by special adjusting devices.

Without describing these devices in detail, we will consider one version of a single-rotor adjustable course indicator, whose theory, in the precessional formulation traditional for gyroscopic compasses, may be found in [13]. The equations of motion of the course indicator, which is assumed to be mounted on a horizontally stabilized platform, the base being stationary relative to the Earth, have the form

$$H\dot{\alpha} - (K + HU\cos\varphi)\beta = 0, \quad H\dot{\beta} + HU\cos\varphi\alpha + K\varepsilon\beta = 0 \tag{4.1}$$

where α and β are the coordinates determining instability, H is the intrinsic angular momentum of the gyroscope, U is the angular velocity of rotation of the Earth, φ is the latitude of the location and K and ε are certain positive constants. The parameters of the gyrocompass are always chosen so as to satisfy the condition $K \gg HU\cos\varphi$. In the content of Eqs (4.1), the torque $K\beta$ imitates the pendulum effect, while the vertical torque $K\varepsilon\,\beta$ guarantees damping of the compass vibrations. The characteristic equation of system (4.1) will be

$$H\lambda^2 + \varepsilon K\lambda + (K + HU\cos\varphi)U\cos\varphi = 0 \tag{4.2}$$

Hence, it follows that if ε and K are positive, the trivial solution of system (4.1) will be asymptotically stable.

The qualitative picture changes considerably if, not confining ourselves to the scope of precession theory, allowance is made for inertial terms in Eqs (4.1). In that case the equations will be

$$\begin{aligned} &J_1\ddot{\alpha} + H\dot{\beta} + HU\cos\varphi\alpha + K\varepsilon\beta = 0 \\ &J_2\ddot{\beta} - H\dot{\alpha} + (K + HU\cos\varphi)\beta = 0 \end{aligned} \tag{4.3}$$

where J_1 and J_2 are the total moments of inertia of the course indicator relative to its suspension axes. The characteristic equation of system (4.3) will be

$$J_1J_2\lambda^4 + [H^2 + KJ_1 + (J_1 + J_2)HU\cos\varphi]\lambda^2 + K\varepsilon H\lambda + (K + HU\cos\varphi)HU\cos\varphi = 0 \tag{4.4}$$

Since Eq. (4.4) contains a linear term in λ but there is no term with λ^3, it follows, unlike the situation in Eq. (4.2), that the trivial solution of system (4.3) is unstable. This instability is a consequence of the non-conservative structure of the vertical component of $K\varepsilon\,\beta$ of the adjusting torque, which, when there are no dissipative forces, leads to divergence of the nutational oscillations.

Indeed, we separate out the non-conservative matrix $P = (K\varepsilon/2)S$ from the matrix of positional forces in system (4.3). In this case asymptotic stability is achievable only if there are dissipative forces in Eqs (4.3). The corresponding equations have the form

$$\begin{aligned} &J_1\ddot{\alpha} + b_1\dot{\alpha} + H(\dot{\beta} + U\cos\varphi\alpha) + K\varepsilon\beta = 0 \\ &J_2\ddot{\beta} + b_2\dot{\beta} - H\dot{\alpha} + (K + HU\cos\varphi)\beta = 0 \end{aligned} \tag{4.5}$$

If we decompose the matrix of positional terms in Eqs (4.5) into its symmetric and skew-symmetric parts, the system turns out to be a special case of the system described by matrix equation (1.1). We have

$$\begin{aligned} &x = \mathrm{col}(\alpha, \beta), \quad J = \mathrm{diag}(J_1, J_2), \quad D = \mathrm{diag}(b_1, b_2) \\ &HG = HS, \quad P = (K\varepsilon/2)S \end{aligned}$$

Thus, the matrices G and P have the structures of matrices (3.1), and the gyrocompass therefore belongs to the class of systems discussed in Section 3.

If one uses the Sommerfeld–Greenhill conceptions, which is very convenient in applied computations, one must put $b_1 = \mu J_1$, $b_2 = \mu J_2$. In that case conditions (3.10), which ensure damping of the nutational oscillations, may be used directly. We have the inequality

$$2H\mu > K\varepsilon \tag{4.6}$$

If $\varepsilon > 0$ and $K > 0$, then, as follows from (4.2), asymptotic stability will also hold in precessional equations (4.1).

Condition (4.6) corresponds to the similar condition (4.4) in [9] (provided one puts $Q_1 = \mu A$, $Q_2 = \mu B$ there), which was obtained for a single-rotor gyrocompass of the pendulum type.

Inequality (4.6) yields a lower bound for the dissipation coefficient μ for which Eqs (4.5) will be asymptotically stable. This bound does not impose any serious restrictions on the value of μ and is generally ensured, with a considerable margin, by the slight friction always present in the suspension elements of the system. In justifying the change to precessional equations (4.1), however, condition (4.6) must be specifically stipulated.

Condition (4.6) may be derived by applying the Hurwitz criterion to the characteristic equation

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0 \tag{4.7}$$

of system (4.5), where

$$\begin{aligned} &a_0 = J_1 J_2, \quad a_1 = 2\mu J_1 J_2 \\ &a_2 = [H^2 + KJ_1 + (J_1 + J_2)HU\cos\varphi + \mu^2 J_1 J_2] \\ &a_3 = HK\varepsilon + [KJ_1 + (J_1 + J_2)HU\cos\varphi]\mu \\ &a_4 = (K + HU\cos\varphi)HU\cos\varphi \end{aligned} \tag{4.8}$$

As applied to Eq. (4.7), since the coefficients of the latter are positive, the Hurwitz conditions reduce to an inequality

$$\Gamma \equiv a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4 > 0 \tag{4.9}$$

Taking expressions (4.8) into consideration, we deduce from (4.9) that

$$\Gamma = H^2 K\varepsilon(2\mu H - K\varepsilon) + \Delta > 0 \tag{4.10}$$

where

$$\begin{aligned} &\Delta = \{2H^2[KJ_1 + (J_1 + J_2)HU\cos\varphi] + KJ_1^2(K + 2\nu HU\cos\varphi) - (J_1 - J_2)^2(HU\cos\varphi)^2\}\mu^2 + \\ &+ 2\varepsilon J_1 J_2 K\mu^3 + 2J_1 J_2[KJ_1 + (J_1 + J_2)HU\cos\varphi]\mu^4, \quad \nu = (J_1 - J_2)J_1^{-1} \end{aligned} \tag{4.11}$$

Since we have assumed that the damping coefficient μ is small and the intrinsic angular momentum H large, the added term Δ, which is the sum of the terms of order μ^2 and higher, cannot affect the sign of the quantity (4.10). The latter is determined in this care by the first terms in (4.10), which leads to condition (4.6)

Thus, direct application of condition (3.10), leading to inequality (4.6), turns out in this case to be much simpler than the use of the Hurwitz criterion with its fairly cumbersome computations, which lead to expressions (4.10) and (4.11).

5. A FOUR-GYROSCOPE GYROHORIZON

In the light of the theory presented in Sections 1–3, we will now consider a version of a force gyrohorizon with control of the radial-correction type. A description of such devices, as well as their theory, limited to the framework of the precessional formulation, may be found in [13]. Based on the complete equations (with inertial terms incorporated), Lyapunov's direct method has been used to investigate the stability of a four-gyroscope gyrohorizon for the case of a fixed base [14].

The system in question is an astatic platform in gimbals stabilized in the horizon by four identical gyroscopes whose cases have vertical axes. Each pair of gyroscopes is linked by anti-parallelograms, which make them rotate in the plane of the platform at the same angles on opposite sides. The motion of the platform is controlled by a special correction system which ensures that adjusting torques are generated relative to the axes of the platform and the gyroscope cases.

The equations of motion of the platform, assuming that it is mounted on a base revolving at constant angular velocity ω relative to the vertical, may be written as follows:

$$\begin{aligned} &J_1\ddot{x}_1 + b_1\dot{x}_1 + 2H\dot{x}_2 + 2H\omega x_3 + s_1 x_2 = 0 \\ &J_2\ddot{x}_2 + b_2\dot{x}_2 - 2H\dot{x}_1 + 2H\omega x_4 - s_2 x_1 = 0 \\ &J_2\ddot{x}_3 + b_2\dot{x}_3 + 2H\dot{x}_4 + 2H\omega x_1 + s_2 x_4 = 0 \\ &J_3\ddot{x}_4 + b_3\dot{x}_4 - 2H\dot{x}_3 + 2H\omega x_2 - s_1 x_3 = 0 \end{aligned} \tag{5.1}$$

where x_1 and x_4 are the angles by which the platform deviates from the plane of the horizon in perturbed motion of the system, x_2 and x_3 are the angles by which each pair of gyroscopes deviates from the vertical axes of their cases, b_1, b_2 and b_3 are the coefficients of viscous friction in the suspension of the platform and the gyroscopes, H is the intrinsic angular momentum of each of the four gyroscopes, s_1 and s_2 and positive coefficients of proportionality in the adjusting torques and J_1, J_2, and J_3 are the intrinsic total moments of inertia of the system about the axes of the platform and the vertical axes of the gyroscope cases.

Decomposing the matrix of positional terms in system (5.1) into symmetric and skew-symmetric parts, we reduce the system to the form covered by the initial matrix equation (1.1), where in this case we must assume that

$$
\begin{gathered}
x = \operatorname{col}(x_1, x_2, x_3, x_4), \quad J = \operatorname{diag}(J_1, J_2, J_2, J_3) \\
D = \operatorname{diag}(b_1, b_2, b_2, b_3), \quad HG = 2H \operatorname{diag}(S, S), \quad P = s \operatorname{diag}(S, S) \\
\Pi = \begin{Vmatrix} mT & 2H\omega E \\ 2H\omega E & -mT \end{Vmatrix}, \quad T = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} \\
s = \frac{1}{2}(s_1 + s_2), \quad m = \frac{1}{2}(s_1 - s_2)
\end{gathered}
\tag{5.2}
$$

Comparing the expressions obtained for G and P with representations (3.1), we see that the gyrohorizon under consideration belongs to the class of systems considered in Section 3. Setting $L = \|l_{ij}\|_1^4$, we obtain Eq. (3.2) in the form

$$
\| \dot{l}_{ij} \|_1^4 = \operatorname{diag}\left(\begin{Vmatrix} 0 & -b_1^{-1}s \\ b_2^{-1}s & 0 \end{Vmatrix}, \begin{Vmatrix} 0 & -b_2^{-1}s \\ b_3^{-1}s & 0 \end{Vmatrix} \right) \| l_{ij} \|
\tag{5.3}
$$

Assuming that the solutions of Eq. (5.3) satisfy the condition $L(0) = E$, we arrive at the Lyapunov matrix $L = \operatorname{diag}(L_1, L_2)$. The components L_1 and L_2 may be found from the general representation (3.5), if we takes $q = 2$ and use the notation in Eqs (5.1).

If, following the Sommerfeld–Greenhill conceptions, one puts $b_1 = \mu J_1$, $b_2 = \mu J_2$, $b_3 = \mu J_3$, then conditions (3.10) may be used to derive a lower bound for the coefficient μ, which will guarantee damping of the nutational oscillations. Using the notation (5.2), this gives

$$
\mu > (s_1 + s_2)/(4H)
\tag{5.4}
$$

In the case of fixed base ($\omega \equiv 0$), when system (5.1) splits into two independent systems in (x_1, x_2) and (x_3, x_4), the validity of condition (5.4) implies that the gyrohorizon is asymptotically stable.

If $\omega \neq 0$, and accordingly $\Pi \neq 0$ in (5.1), further study of the precessional system obtained from (5.1) is necessary, as noted in Section 3. This has been done for the case $b_3 = b_1$ [13] using the Hurwitz criterion; in terms of the notation for system (5.1), the result is the inequality

$$
s_1 s_2 > b_1 b_2 \omega^2
\tag{5.5}
$$

The simultaneous validity of conditions (5.4) and (5.5) guarantees that the system with $\omega \neq 0$ will be asymptotically stable.

Inequality (5.5) agrees with the conditions for the matrix $W_1 = JW$, where W is defined as in the last formula of (2.8), to be positive-definite; this is obvious if one notes that the similarity transformation (2.8) does not change the eigenvalues of the matrices V and W in Eqs (2.7). In that case, according to the method used to derive the precessional equations, one must retain in the matrix W_1 for this case only those terms that contain the intrinsic angular momentum H of each of the four gyroscopes as a factor. We have

$$
W_1 = 2H \begin{Vmatrix} b_2^{-1}s & 0 & \omega & 0 \\ 0 & b_1^{-1}s & 0 & \omega \\ \omega & 0 & b_3^{-1}s & 0 \\ 0 & \omega & 0 & b_2^{-1}s \end{Vmatrix}
\tag{5.6}
$$

Applying Sylvester's criterion to the symmetric matrix (5.6) we obtain in the conditions for it to be positive-definite. Putting $b_3 = b_1$ in the matrix, we obtain the inequality

$$(s_1 + s_2)^2 > 4b_1b_2\omega^2 \tag{5.7}$$

Since, if $s_1 \neq s_2$, it is always true that $(s_1 + s_2)^2 > 4s_1s_2$, it follows that if inequality (5.5) holds, the same is all the more true of condition (5.7)

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